ON A GAME PROBLEM OF PROGRAM CONTROL

PMM Vol. 35, №1, 1971, pp. 132-137
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(Received July 29, 1970)

It is shown that the solvability of the problem of the program maximin [1] of the time of transfer of a linear system with constant coefficients does not generally imply the solvability of the analogous problem in inverse time.

It is shown that the technique of time inversion commonly employed in the theory of controlled processes can be used in the conflict situations in question only if additional conditions are imposed.

1. The question posed below is answered in the present paper.

We are given the linear system

$$dx / dt = Ax + u - v \tag{1.1}$$

Here A is a constant ($n \times n$)-matrix; x is an n-dimensional vector column; u and v are the controlling parameters subject at all instants to the restrictions

$$u(t) \in U, \qquad v(t) \in V,$$

where U and V are bounded convex closed sets of dimension not higher than n.

System (1.1) corresponds in the inverse time $\tau = -t$ to the system

$$dx / d\tau = -Ax - u + v \tag{1.2}$$

The controlling parameters u, v of the system at each instant τ belong to the sets U, V, respectively.

Let $u^{(1)}(\cdot)$, $v^{(1)}(\cdot)$, $(u^{(2)}(\cdot)$, $v^{(2)}(\cdot))$ be arbitrary measurable programs $u(t)$, $v(t)$ $(u(\tau)$, $v(\tau))$ specified over an infinite time interval $0 \leqslant t < \infty$ $(0 \leqslant \tau < \infty)$; for any t (τ) these programs satisfy the conditions

$$u(t) \in U, \ v(t) \in V \qquad (u(\tau) \in U, \ v(\tau) \in V)$$

Let us isolate two points $x^{(1)}$ and $x^{(2)}$ in the phase space. By

$$T^{(1)}[u^{(1)}(\cdot), v^{(1)}(\cdot)] \qquad (T^{(2)}[u^{(2)}(\cdot), v^{(2)}(\cdot)])$$

we denote the earliest instant by which system (1.1), (1.2) operating under the programs $u^{(1)}(\cdot)$, $v^{(1)}(\cdot)$, $(u^{(2)}(\cdot)$, $v^{(2)}(\cdot))$ can be transferred from the point $x^{(1)}$ $(x^{(2)})$ to the point $x^{(2)}$ $(x^{(1)})$. If such transfer cannot be effected by these programs, we set

$$T^{(1)}[u^{(1)}(\cdot), v^{(1)}(\cdot)] = \infty \qquad (T^{(2)}[u^{(2)}(\cdot), v^{(2)}(\cdot)] = \infty)$$

We stipulate that

$$T^{(i)} = \sup_v \min_u T^{(i)}[u^{(i)}(\cdot), v^{(i)}(\cdot)] \qquad (i = 1, 2)$$

(the exact upper bound (minimum) is taken over all the possible programs $v^{(i)}(\cdot)$, $(u^{(i)}(\cdot))$).

Question: does the finiteness of the time $T^{(i)}$ $(i = 1, 2)$ imply the finiteness of the time $T^{(j)}$ $(j = 2, 1;\ j \neq i)$?

In the general case (without additional assumptions), the answer to this question is negative. This can be demonstrated by considering a second-order system.

2. Consider the second-order system

$$dx_1 / dt = x_2, \qquad dx_2 / dt = u - v \tag{2.1}$$

with the following restrictions imposed on the controlling parameters:

$$|u| \leqslant \mu, \; v = w + c, \; |w| \leqslant \nu, \; \nu < \mu, \; c = \text{const}, \quad \mu < c < \mu + \nu \tag{2.2}$$

In inverse time system (2.1) corresponds to the system

$$dx_1 / d\tau = -x_2, \qquad dx_2 / d\tau = -u + v \tag{2.3}$$

under restrictions (2.2).

Let us set $x^{(2)} = 0$. We denote the point $x^{(1)}$ by m. We shall assume from now on that the point m belongs to the set

$$A = \left\{x : x_2 > 0, \; -\frac{x_2^2}{2(\mu - \nu + c)} + \frac{(\mu - \nu)(\mu + \nu - c)^2 x_2^2}{4(\mu + \nu + c)^2 (\mu - \nu + c)^2} > \right.$$
$$\left. > x_1 > -\frac{x_2^2}{2(\mu - \nu + c)}\right\} \tag{2.4}$$

Let us show that for any point $m \in A$ we have

$$T^{(1)} = \infty, \qquad T^{(2)} < \infty$$

3. Statement 3.1. The time $T^{(1)} = \infty$ for any point m from the set A.

Proof. We stipulate that

$$t^\circ = \frac{x_{2m}}{\mu - \nu + c}$$

In system (2.1) let us replace u by an arbitrary program $u(t)$ ($|u(t)| \leqslant \mu$) and v by the program

$$v^\circ(t) = \begin{cases} -\nu + c, & \text{if} \quad 0 \leqslant t < t^\circ \\ \nu + c, & \text{if} \quad t^\circ \leqslant t \end{cases}$$

Moreover, let system (2.1) be at the point m at the initial instant $t = 0$.

The Cauchy formula

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1m} \\ x_{2m} \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & t - s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ u(s) - v^\circ(s) \end{pmatrix} ds$$

with allowance for (2.2), (2.4) implies the following estimates of the position of system (2.1):

if $0 \leqslant t < t^\circ$, $x_2(t) > 0$,

$$x_1(t^\circ) \geqslant \frac{x_{2m}^2}{2(\mu - \nu + c)} + x_{1m}, \quad x_2(t^\circ) \geqslant 0$$

if $x_2(t) \geqslant 0$, $t > t^\circ$,

$$x_1(t) > \frac{x_{2m}^2}{2(\mu - \nu + c)} + x_{1m}$$

if $x_2(t) < 0$, $t > t^\circ$,

$$x_1(t) \geqslant \frac{x_{2m}^2}{2(\mu - \nu + c)} + x_{1m} - \frac{x_2^2(t)}{2(-\mu + \nu + c)}$$

From these estimates we see that system (2.1) does not reach the origin for any $t \geqslant 0$. Since the program $u(t)$ is arbitrary, this implies that the time $T^{(1)} = \infty$. the statement has been proved.

4. Before proving the fact that the time $T^{(2)} < \infty$ for any point $m \in A$, let us carry out certain ancillary constructions (see Fig. 1).

Let us have the motions of system (2.1) emerging from the point m for $u = -\mu$, $v = -\nu + c$ and for $u = \mu$, $v = \nu + c$. The trajectory of the first motion is described by the equation

$$x_1 = \frac{x_{2m}^2 - x_2^2}{2(\mu - \nu + c)} + x_{1m}, \qquad x_2 \leqslant x_{2m} \tag{4.1}$$

and of the second motion by the equation

$$x_1 = \frac{x_{2m}^2 - x_2^2}{2(-\mu + \nu + c)} + x_{1m}, \qquad x_2 \leqslant x_{2m} \tag{4.2}$$

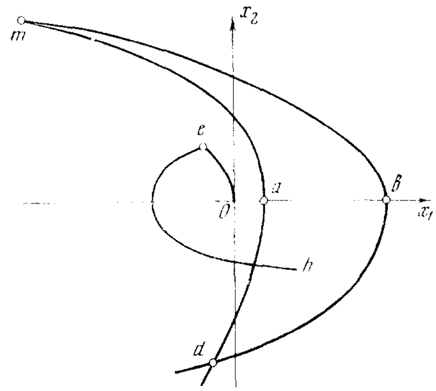

Fig. 1

We denote the points of intersection of curves (4.1) and (4.2) with the axis x_1 by a and b, respectively

$$x_{1a} = \frac{x_{2m}^2}{2(\mu - \nu + c)} + x_{1m}, \qquad x_{1b} = \frac{x_{2m}^2}{2(-\mu + \nu + c)} + x_{1m}$$

From point a we construct the motion of system (2.1) for $u = -\mu$, $v = \nu + c$, and from point b its motion for $u = -\mu$, $v = -\nu + c$. The trajectories of these motions, i.e.

$$x_1 = \frac{x_{2m}^2}{2(\mu - \nu + c)} + x_{1m} - \frac{x_2^2}{2(\mu + \nu + c)}, \qquad x_2 \leqslant 0$$

$$x_1 = \frac{x_{2m}^2}{2(-\mu + \nu + c)} + x_{1m} - \frac{x_2^2}{2(\mu - \nu + c)}, \qquad x_2 \leqslant 0$$

intersect at the point d,

$$x_{1d} = \frac{x_{2m}^2 [2\nu c - \mu(\mu - \nu + c)]}{2\nu [c^2 - (\mu - \nu)^2]} + x_{1m}$$

$$x_{2d} = -x_{2m} \left(\frac{(\mu - \nu)(\mu + \nu + c)}{\nu(-\mu + \nu + c)} \right)^{1/2}$$

S t a t e m e n t 4.1. System (2.3) can be transferred from any initial position on the curve mad to the point m with any program $v(\tau) = w(\tau) + c\,(|w(\tau)| \leqslant \nu)$ in a time not exceeding some number t_*.

The proof can be broken up into two stages. First we consider transfer of system (2.3) from the curve mad to the curve mb, and then from the curve mb to the point m.

Let system (2.3) lie at some point on the curve mad at the initial instant. Let us set $u = -\mu$. Then for any program $v(\tau)$ we have

$$dx_1 / d\tau = -x_2, \quad \mu + \nu + c \geqslant dx_2 / d\tau = \mu + v(\tau) \geqslant \mu - \nu + c > 0$$

Hence, the motion occurs between the curves mad, bd and reaches the curve mb in a finite time. The time of transfer from the curve mad to the curve mb is bounded above by the number

$$t_1 = \frac{x_{2m} - x_{2d}}{\mu - \nu + c}$$

Let system (2.3) lie on the curve mb at the initial instant. Then for any program $v(\tau)$ the program

$$u(\tau) = \mu - \nu - c + v(\tau)$$

ensures motion of system (2.3) along the curve mb in the direction of the point m, and the velocity of this motion does not depend on the program $v(\tau)$. The time of motion to the point m is bounded above by the number

$$t_2 = \frac{x_{2m}}{-\mu + \nu + c}$$

Thus, transfer of system (2.3) from the curve mad to the point m for any program $v(\tau)$ can be completed in a time not larger than the number $t_* = t_1 + t_2$. The statement has been proved.

Let us set

$$\alpha = \frac{x_{2m}}{\mu - \nu + c} - \left[\frac{4\nu}{(\mu + c)^2 - \nu^2}\left(x_{1m} + \frac{x_{2m}^2}{2(\mu - \nu + c)}\right)\right]^{1/2}$$

$$v_* = x_{2m} / \alpha - \mu$$

Since the point $m \in A$, it follows that the quantities α and v_* satisfy the inequalities

$$\frac{x_{2m}}{\mu - \nu + c} > \alpha > \frac{x_{2m}}{\mu - \nu + c} - \frac{x_{2m}(\mu + \nu - c)}{(\mu + \nu + c)(\mu - \nu + c)} \times$$

$$\times \sqrt{\frac{(\mu - \nu)\nu}{(\mu + c)^2 - \nu^2}} > \frac{2cx_{2m}}{(\mu + c)^2 - \nu^2} > \frac{x_{2m}}{2\mu}$$

$$\mu > v_* > -\nu + c$$

S t a t e m e n t 4.2. System (2.3) can be transferred from the origin to the curve ma in a time not larger than α for any program $v(\tau) = w(\tau) + c$ $(|w(\tau)| \leqslant \nu)$ with its mean value on the segment $0 \leqslant \tau \leqslant \alpha$,

$$\frac{1}{\alpha}\int_0^\alpha v(\tau)\,d\tau \geqslant v_*$$

P r o o f. Let system (2.3) be at the origin at the instant $\tau = 0$. Then let it move under programs $v(\tau)$ and $u(\tau) = -\mu$. For any $\tau \geqslant 0$ we have

$$dx_2 / d\tau = \mu + v(\tau) > 0$$

Let us denote by β the instant at which system (2.3) reaches the straight line $x_2 = x_{2m}$; $\beta \leqslant \alpha$, since by hypothesis

$$\frac{1}{\alpha}\int_0^\alpha v(\tau)\,d\tau \geqslant v_* \qquad \left(\beta = \alpha \quad \text{for} \quad \frac{1}{\alpha}\int_0^\alpha v(\tau)\,d\tau = v_*\right)$$

By v^* we denote the mean value of the program $v(\tau)$ over the interval $0 \leqslant \tau \leqslant \beta$,

$$v^* = \frac{1}{\beta}\int_0^\beta v(\tau)\,d\tau = \frac{x_{2m}}{\beta} - \mu$$

Let us introduce the ancillary program

$$v^\circ(\tau) = \begin{cases} \nu + c, & \text{if} \quad 0 \leqslant \tau < \beta^\circ, \\ -\nu + c, & \text{if} \quad \beta^\circ \leqslant \tau \end{cases}$$

$$\beta^\circ = \frac{x_{2m} - \beta(\mu - \nu + c)}{2\nu} < \beta$$

The mean value of the program $v^\circ(\tau)$ over the segment $0 \leqslant \tau \leqslant \beta$ is v^*.

The position of system (2.3) at the instant $\tau = \beta$ can be estimated as follows:

$$x_2(\beta) = x_{2m}, \quad x_1(\beta) = \int_0^\beta (\tau - \beta)[\mu + v(\tau)]\,d\tau \geqslant \int_0^\beta (\tau - \beta)[\mu + v^\circ(\tau)]\,d\tau =$$

$$= \frac{(\mu + c)^2 - \nu^2}{4\nu}\left(\frac{x_{2m}}{\mu - \nu + c} - \beta\right)^2 - \frac{x_{2m}^2}{2(\mu - \nu + c)} \geqslant$$

$$\geqslant \frac{(\mu + c)^2 - \nu^2}{4\nu}\left(\frac{x_{2m}}{\mu - \nu + c} - \alpha\right)^2 - \frac{x_{2m}^2}{2(\mu - \nu + c)} = x_{1m}.$$

Since for $\tau > 0$ we have

$$dx_1 / d\tau = -x_2 < 0$$

these estimates imply that system (2.3) intersects the curve ma for $\tau \leqslant \beta \leqslant \alpha$. The statement has been proved.

Statement 4.3. System (2.3) can be transferred from the origin to the curve ad in a time not larger than α under any program $v(\tau) = w(\tau) + c$ ($|w(\tau)| \leqslant \nu$) with its mean value on the segment $0 \leqslant \tau \leqslant \alpha$ given by

$$\frac{1}{\alpha}\int_0^\alpha v(\tau)\,d\tau < v_*$$

Proof. Let us introduce the ancillary program

$$v^\circ(\tau) = \begin{cases} \nu + c, & \text{if } 0 \leqslant \tau < \alpha^\circ \\ -\nu + c, & \text{if } \alpha^\circ \leqslant \tau \end{cases} \qquad \alpha^\circ = \frac{x_{2m} - \alpha(\mu - \nu + c)}{2\nu} < \alpha$$

The mean value of the program $v^\circ(\tau)$ over the segment $0 \leqslant \tau \leqslant \alpha$ is v_*.

The motion $x^\circ(\tau)$ (trajectory Oeh in Fig.1) of system (2.3) under the programs $v^\circ(\tau)$, $u(\tau) = \mu$ beginning at the origin at the instant $\tau = 0$ is described by the following equations for $\tau \geqslant \alpha^\circ$:

$$x_1^\circ(\tau) = (\mu - \nu - c)(\alpha^\circ)^2/2 + (\mu - \nu - c)\alpha^\circ(\tau - \alpha^\circ) + \\ + (\mu + \nu - c)(\tau - \alpha^\circ)^2/2 \qquad (4.3)$$

$$x_2^\circ(\tau) = (-\mu + \nu + c)\alpha^\circ - (\mu + \nu - c)(\tau - \alpha^\circ)$$

It is easy to verify that for any point $m \in A$ this motion intersects the curve ad in the interval $\alpha^\circ < \tau \leqslant \alpha$.

Now let us consider the case of an arbitrary program $v(\tau)$ whose mean value in the segment $0 \leqslant \tau \leqslant \alpha$ is smaller than v_*. Again we set $u(\tau) = \mu$. The motion $x(\tau)$ of system (2.3) under the programs $v(\tau)$ and $u(\tau)$ lies at any $\tau \geqslant 0$ strictly to the right of the trajectory eh of motion (4.3). Bearing in mind that

$$x_2(\alpha) < x_2^\circ(\alpha)$$

we infer from this that the motion $x(\tau)$ intersects the curve ad for $\tau \leqslant \alpha$. The statement has been proved.

Statements 4.1–4.3 imply that $T^{(2)} < \infty$ for any point $m \in A$.

The author is grateful to N.N. Krasovskii for his comments and useful remarks.

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Translated by A.Y.